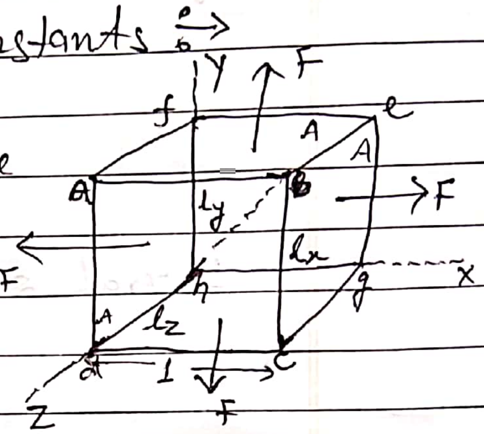
D-1 Physics (Hons)

Relation among Elastic Constants :-
(Y, K, η, σ)



Consider a cube of unit volume
i.e $l_x = 1$, $l_y = 1$, $l_z = 1$
volume $V = 1 \times 1 \times 1 = 1$ unit
The area(A) of each face is 1 unit.
we also consider the applied force (F) is 1 unit.

$Y \rightarrow$ Young's modulus, $K \rightarrow$ Bulk modulus
$\eta \rightarrow$ Modulus of rigidity, $\sigma \rightarrow$ Poisson ratio

1 Relation among (Y, K and σ):

Bulk modulus is define as

$$K = \frac{\text{Volume stress}}{\text{Volume strain}}$$

$$\text{volume stress} = \frac{F}{A}$$

but $F = 1$ unit, $A = 1$ unit.

$$\text{volume stress} = \frac{1}{1} = 1$$

$$\text{volume strain} = \frac{\Delta V}{V}$$

$V = 1$ unit.

$\Delta V = ?$

$$\text{volume strain} = \frac{\Delta V}{1} = \Delta V$$

Now $$K = \frac{1}{\Delta V} \quad \text{--- (1)}$$

Longitudinal strain (α) is define as

$$\alpha = \frac{\Delta l}{l}$$

$$l = 1 \text{ unit}$$

$$\alpha = \frac{\Delta l}{1} = \Delta l$$

Lateral strain (β) is define as:

$$\beta = \frac{\Delta l}{l} = \frac{\Delta l}{1} = \Delta l$$

If we apply the force along x direction then the new length of the cube along-x will be

$$l_x' = 1 + \alpha - \beta - \beta$$

$$l_x' = 1 + \alpha - 2\beta$$

Similarly if we apply force along Y and z-direction then

$$l_y' = 1 + \alpha - 2\beta$$

$$l_z' = 1 + \alpha - 2\beta$$

Now the change in Volume will be

$$\Delta V = V' - V$$

$$V' = l_x' \times l_y' \times l_z' \quad , \quad V = 1 \text{ unit.}$$

$$V' = (1 + (\alpha - 2\beta))^3$$

using Bionomial theorem

$$\Delta V' = 1 + 3(\alpha - 2\beta) \qquad (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

$$V' = 1 + 3(\alpha - 2\beta)$$

$$\Delta V = \cancel{1} + 3(\alpha - 2\beta) - \cancel{1}$$

$$\Delta V = 3(\alpha - 2\beta)$$

$$\Delta V = 3\alpha\left(1 - 2\frac{\beta}{\alpha}\right)$$

Poisson ratio $\sigma = \frac{\beta}{\alpha}$

$$\Delta V = 3\alpha(1-2\sigma) \qquad (\because \sigma = \beta/\alpha)$$

$$K = \frac{1}{\Delta V}$$

$$K = \frac{1}{3\alpha(1-2\sigma)}$$

$$K = \frac{1/\alpha}{3(1-2\sigma)} \quad \text{——— (II)}$$

Young's modulus Y is define as:

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

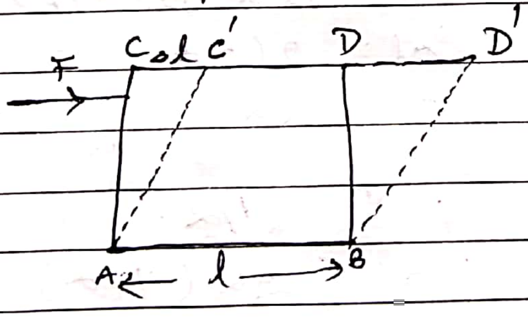
$$Y = \frac{1}{\frac{\Delta l}{l}}$$

$$Y = \frac{1}{\alpha} \quad \text{——— (III)}$$

$$\boxed{K = \frac{Y}{3(1-2\sigma)}}$$

this is the relation among K, Y and σ.

(II) Relation among Y, η and σ :⇒



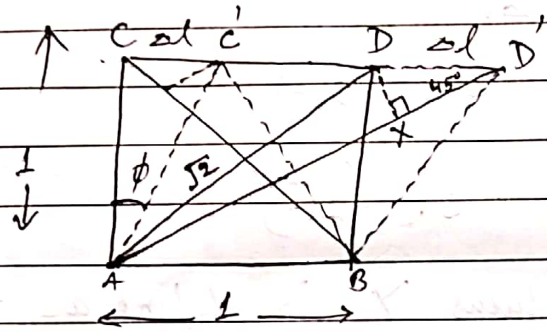
$l = 1$ unit
$A = 1$ unit
$V = 1$ unit
$F = 1$ unit

modulus of rigidity (η) is define as

$$\eta = \frac{\text{Shearing stress}}{\text{Shearing strain}}$$

$$\eta = \frac{F/l^2}{\frac{\Delta l}{l}}$$

$$\eta = \frac{1}{\Delta l} \quad \text{------} \quad (B) \quad \left(\because \begin{array}{l} F = 1 \text{ unit} \\ l = 1 \text{ unit} \end{array} \right)$$



Longitudinal strain $(\alpha) = \frac{\Delta l}{l} = \Delta l$

Lateral strain $= \frac{\Delta D}{D}$

change in length $\cancel{\Delta} XD' = AD \times (\alpha + \beta)$ $\quad (\because AD = AX)$

$\cancel{XD'} = \sqrt{2}(\alpha + \beta)$

$XD' = \sqrt{2}(\alpha + \beta)$

In $\Delta DD'X$

$$\cos 45^\circ = \frac{XD'}{\Delta l}$$

$$\frac{1}{\sqrt{2}} = \frac{XD'}{\Delta l}$$

$$\Delta l = \sqrt{2}\, XD'$$

$$\Delta l = \sqrt{2} \times \sqrt{2}(\alpha + \beta)$$

$$\Delta l = 2(\alpha + \beta)$$

$$\Delta l = 2\alpha\left(1 + \frac{\beta}{\alpha}\right)$$

$$\sigma = \beta/\alpha$$

$$\Delta l = 2\alpha(1 + \sigma)$$

$$\frac{1}{\Delta l} = \frac{1}{2\alpha(1+\sigma)}$$

$$\frac{1}{\Delta l} = \frac{1/\alpha}{2(1+\sigma)}$$

$$\frac{1}{\Delta l} = \eta$$

$$\frac{1}{\alpha} = Y$$

$$\frac{1}{\alpha} = \frac{1/\alpha}{2(1+\sigma)}$$

$$\boxed{\eta = \frac{Y}{2(1+\sigma)}} \quad \text{—— (IV)}$$

This is the relation among η, Y and σ.

(iii) Relation among Y, K and η :→

we already drived

$$K = \frac{Y}{3(1-2\sigma)} \quad \text{—— (III)}$$

$$\eta = \frac{Y}{2(1+\sigma)} \quad \text{—— (IV)}$$

$$K = \frac{Y}{3(1-2\sigma)}$$

$$1-2\sigma = \frac{Y}{3K} \quad \text{—— (V)}$$

$$\eta = \frac{Y}{2(1+\alpha)}$$

$$2+2\alpha = \frac{Y}{\eta} \quad \text{—— (VI)}$$

adding equation (V) and (VI)

$$3 = \frac{Y}{3K} + \frac{Y}{\eta}$$

$$3 = Y\left(\frac{1}{3K} + \frac{1}{\eta}\right)$$

$$\boxed{\frac{3}{Y} = \frac{1}{3K} + \frac{1}{\eta}}$$ Relation among Y, K and η

(iv) Relation among K, η and σ :→

$$K = \frac{Y}{3(1-2\sigma)} \quad \text{—— (iii)}$$

$$\eta = \frac{Y}{2(1+\sigma)} \quad \text{—— (iv)}$$

$$K = \frac{Y}{3(1-2\sigma)}$$

$$Y = 3K(1-2\sigma)$$

Put the value of Y in eqⁿ (iv)

$$\eta = \frac{3K(1-2\sigma)}{2(1+\sigma)}$$

$$2\eta(1+\sigma) = 3K(1-2\sigma)$$

$$2\eta + 2\eta\sigma = 3K - 6K\sigma$$

$$2\eta\sigma + 6K\sigma = 3K - 2\eta$$

$$2\sigma(\eta + 3K) = 3K - 2\eta$$

$$\boxed{\sigma = \frac{3K - 2\eta}{2(\eta + 3K)}}$$ This is the relation among K, η and σ.

or

$$\boxed{K = \frac{(2+2\sigma)\eta}{3(1-2\sigma)}}$$